

## ON HÀJEK - RÉNYI TYPE INEQUALITY AND APPLICATION

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ABSTRACT. In this paper, we establish a new Hájek - Rényi's type inequality and obtain strong law of large numbers (SLLN) for arbitrary random variables. We base our methods on demimartingales and convex functions techniques. We obtain wide extensions of available results, particularly those of Amini et al. [1] and Rao [14] and related SLLN's.

## 1. INTRODUCTION

This paper is concerned with general strong laws of large numbers (SLLN) based on Hájek - Rényi's type inequalities and therein provides sharp generalization of recent forms of such inequalities. To begin with, we remind the seminal result of Hájek - Rényi [8]. Let  $X_1, X_2, \dots$  denote a sequence of random variables defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ ,  $S_0 = 0$ ,  $S_n = \sum_{i=1}^n X_i$  for  $n \geq 1$ .

**Lemma 1.** (*Inequality of Hájek - Rényi*). *If  $\{X_n, n \geq 1\}$  is a sequence of independent random variables with mean zero, and  $(b_n)_{n \geq 1}$  is a non-decreasing sequence of positive real numbers, then for any  $\varepsilon > 0$  and any positive integer  $m \leq n$ ,*

$$\mathbb{P}\left(\max_{m \leq k \leq n} \frac{\sum_{j=1}^k X_j}{b_k} > \varepsilon\right) \leq \varepsilon^{-2} \sum_{j=m+1}^n \mathbb{E}\left(\frac{X_j^2}{b_j^2}\right) + \frac{1}{b_m^2} \sum_{j=1}^m \mathbb{E}(X_j^2).$$

This inequality has been studied and generalized by many authors. The latest literature is given by Liu et al. [11] for negative association random variables, Christofides [2] for demimartingales, Christofides and Vaggelaton [3] for positive and negative association, Rao [16] and Sung [17] and Hu *et al.* [9] for associated random variables and Wang *et al.* [19] for demimartingales which extended [2]. Recent developments on the topic can be found in Rao [15]. It is also worth mentioning that highly sophisticated forms of such inequalities and related SLLN's are

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available for random fields (see Noszály et al. [13], Farzkas et al. [5], Hamallah et al. [7], etc.).

Our departure point is the Hájek - Rényi type inequality of Amini et al. [1], established for arbitrary random variables when the second moments of the  $X_n$ 's are finite. Such results do not apply to stable random variables of parameter  $1 \leq \alpha < 2$  for example.

This motivated us to extend the preceding Hájek - Rényi type of inequalities to a more general one. We would get more general SLLN's and would be able to handle cases the former could not do, as the one mentioned above.

To achieve our objectives we call on demimartingales techniques and convex functions properties and provide a generalization of the inequality of Rao [14] as a basis of our main inequality result in Theorem 1 below. Next, we derive from it our general SLLN in Theorem 2, which finally resulted in a considerable generalizations of the SLLN's of [1].

The paper is organized as follows. In Section 2, we present some definitions and lemmas which we need to prove our main result. In the section 3, we establish a Hájek - Rényi's inequality for arbitrary random variables. As a consequence, we obtain a strong law of large numbers for arbitrary random variables. We conclude the paper by applications and comparison remarks with former results.

Whenever needed, any sequence of random variables used in the paper may be required to be  $\mathcal{F}_n$ -adapted where  $(\mathcal{F}_n, n \geq 0)$  is some filtration on  $(\Omega, \mathcal{F}, P)$ . We also adopt the classical notation  $X_n^+ = \max(0, X_n)$  and  $X_n^- = \max(0, -X_n)$ ,  $n = 1, 2, \dots$

## 2. DEFINITIONS AND SOME REMINDERS

We begin to remind the notion of demimartingale introduced by Newmann and Wright [12].

**Definition 1.** A  $\mathcal{F}_n$ -adapted sequence of random variables  $\{T_n, n \geq 1\}$  in  $L^1(\Omega, \mathcal{F}, \mathbb{P})$  is called a demimartingale if, for every  $j \geq 1$ , for every componentwise nondecreasing function  $g(x)$  of  $x \in \mathbb{R}^j$ ,

$$(2.1) \quad \mathbb{E}((T_{j+1} - T_j) g(T_1, \dots, T_j)) \geq 0.$$

If further the function  $g$  in (2.1) are required to be nonnegative (resp. non positive), then the sequence is called a demisubmartingale (resp., a demisupermartingale).

Let us make some immediate remarks. First a  $\mathcal{F}_n$ -martingale  $T_n$  is a demimartingale. This can be seen by noting that

$$\begin{aligned} \mathbb{E}\left((T_{j+1} - T_j)g(T_1, \dots, T_j)\right) &= \mathbb{E}\left[\mathbb{E}\left((T_{j+1} - T_j)g(T_1, \dots, T_j)\right) \middle| \mathcal{F}_j\right] \\ &= \mathbb{E}\left[g(T_1, \dots, T_j)\mathbb{E}\left((T_{j+1} - T_j) \middle| \mathcal{F}_j\right)\right] = 0 \end{aligned}$$

by using the martingale property of the process  $\{T_n, \mathcal{F}_n, n \geq 1\}$ . Similarly it can be seen that every  $\mathcal{F}_n$ -submartingale is a demisubmartingale. However a demisubmartingale need not be a  $\mathcal{F}_n$ -submartingale as showed by [6], even the filtration is the natural one.

We now recall some connexions between convex functions and demimartingales.

**Lemma 2.** (see Lemma 2.1 and Corollary 2.1 in [2])

i) Let  $T_1, T_2, \dots$ , be a demisubmartingale (or a demimartingale) and  $\phi$  a nondecreasing convex function such that  $\phi(T_i) \in L_1$ ,  $i \geq 1$ . Then  $\phi(T_1), \phi(T_2), \dots$ , is a demisubmartingale.

ii) If  $\{T_n, n \geq 1\}$  is a demimartingale, then  $\{T_n^+, n \geq 1\}$  is a demisubmartingale et  $\{T_n^-, n \geq 1\}$  is a demisubmartingale.

Rao introduced the following inequality that we are going to generalize.

**Lemma 3.** (see Rao [14]). Let  $\{T_n, n \geq 1\}$  be a demisubmartingale and  $\phi(\cdot)$  be a nonnegative nondecreasing convex function such that  $\phi(T_o) = 0$ . Let  $\chi(b)$  be a positive nondecreasing function of  $b > 0$  and let  $0 = b_o < b_1 \leq \dots \leq b_n$ . Then

$$\mathbb{P}(\phi(T_k) \leq \chi(b_k), 1 \leq k \leq n) \geq 1 - \sum_{k=1}^n \frac{\mathbb{E}[\phi(T_k)] - \mathbb{E}[\phi(T_{k-1})]}{\chi(b_k)}.$$

We also need this result to derive general SLLN's.

**Lemma 4.** (See Theorem 2.4 in Tórnács-Libor [18]). Let  $\{\alpha_k, k \in N\}$  be a sequence of non-negative real numbers,  $r > 0$ , and  $\{b_k, k \in N\}$  a nondecreasing unbounded sequence of positive real numbers. Assume that  $\sum_{k=1}^{\infty} \alpha_k b_k^{-r} < \infty$  and that there exists  $c > 0$  such that for any

$n \in N$  and any  $\varepsilon > 0$

$$\mathbb{P}(\max_{k \leq n} |S_k| \geq \varepsilon) \leq c\varepsilon^{-r} \sum_{k=1}^n \alpha_k.$$

Then

$$\lim_{n \rightarrow +\infty} \frac{S_n}{b_n} = 0, \text{ a.s.}$$

### 3. OUR RESULTS

Before we state our results, let us make this simple remark : there exist two demisubmartingales

$$(3.1) \quad \{u_n, n \geq 1\} \text{ and } \{v_n, n \geq 1\}$$

such that

$$(3.2) \quad \forall n \geq 1, S_n \leq u_n + v_n \text{ a.s.}$$

To see that it suffices to set  $u_n = \sum_{i=1}^n X_i^+$  and  $v_n = \sum_{i=1}^n X_i^-$ . Then (3.2) is evident. Finally, each of these sequence is a  $\mathcal{F}_n$ -submartingale and then a demisubmartingale.

We use the demisubmartingales  $u_n$  and  $v_n$  in our general Hjek - Rnyi's type of inequality below.

**Theorem 1.** *Let  $\{X_n, n \geq 1\}$  be an arbitrary sequence of a.s finite random variables. Let  $\phi(\cdot)$  be a nonnegative nondecreasing convex function such that  $\phi(S_0) = 0$  and there exist a positive number real  $K$  such that for all  $(x, y) \in \mathbb{R}^2$ ,*

$$\phi(x + y) \leq K[\phi(x) + \phi(y)]$$

*Let  $\chi(b)$  be a positive non decreasing function of  $b > 0$  and  $0 = b_0 < b_1 \leq b_2 \leq \dots \leq b_n$  and finally set  $A_n = (\phi(S_k) \leq \chi(b_k), 1 \leq k \leq n)$ ,  $n \geq 1$ . Then for any  $n \geq 1$ ,*

$$\mathbb{P}(A_n) \geq 1 - 2K \sum_{k=1}^n \frac{\mathbb{E}[\phi(u_k) + \phi(v_k)] - \mathbb{E}[\phi(u_{k-1}) + \phi(v_{k-1})]}{\chi(b_k)}$$

*whenever the right member is well-defined and where  $\{u_k, k \geq 1\}$  and  $\{v_k, k \geq 1\}$  are two demisubmartingales defined in (3.1)*

*Proof.* By (3.2),

$$S_k \leq u_k + v_k, \quad 1 \leq k \leq n, \text{ a.s.}$$

and then

$$\phi(S_k) \leq \phi(u_k + v_k) \leq K[\phi(u_k) + \phi(v_k)] \text{ a.s.}$$

Next,

$$\begin{aligned}
\mathbb{P}(A_n^c) &= \mathbb{P}\left(\max_{1 \leq k < n} \frac{\phi(S_k)}{\chi(b_k)} \geq 1\right) \\
&\leq \mathbb{P}\left(\max_{1 \leq k < n} \frac{K\phi(u_k)}{\chi(b_k)} \geq 1/2\right) + \mathbb{P}\left(\max_{1 \leq k < n} \frac{K\phi(v_k)}{\chi(b_k)} \geq 1/2\right) \\
&\leq 2K \sum_{k=1}^n \frac{\mathbb{E}[\phi(u_k) + \phi(v_k)] - \mathbb{E}[\phi(u_{k-1}) + \phi(v_{k-1})]}{\chi(b_k)},
\end{aligned}$$

whenever the right member is well-defined.  $\square$

The above result is a generalization of the result of Rao [14] established for demisubmartingales to a sequence of arbitrary random variables.

**Theorem 2.** *Let  $\{X_n, n \geq 1\}$  be an arbitrary a.s. finite random variables and  $\phi(\cdot)$  be a nonnegative nondecreasing convex function such that  $\phi(0) = 0$ . Let  $\chi(b)$  be a positive nondecreasing function of  $b > 0$  such that  $\chi(b) \rightarrow \infty$  as  $b \rightarrow \infty$  and let  $(b_n)_{n \geq 1}$  a nondecreasing and unbounded sequence of positive real numbers. Further suppose the following series is well-defined and satisfies*

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}[\phi(u_k) + \phi(v_k)] - \mathbb{E}[\phi(u_{k-1}) + \phi(v_{k-1})]}{\chi(b_k)} < \infty,$$

where  $u_k$  and  $v_k$  are demisubmartingales defined in 3.2. Then

$$\frac{\phi(S_n)}{\chi(b_n)} \rightarrow 0, \text{ a.s. as } n \rightarrow \infty.$$

*Proof.* The result follows immediately by Lemma 4. We omit the details.  $\square$

## 4. APPLICATIONS AND COMPARISON

**4.1. Comparison with former results.** Our result is to be compared with the following results of Amini et al. [1] for arbitrary random variables.

**Theorem 3.** *(See Amini and Bozorgnia, A [1]) Let  $\{X_n, n \geq 1\}$  be a sequence of random variables with  $\mathbb{E}(X_n) = 0$ ,  $\sigma_n^2 = \text{Var}(X_n) =$*

$\mathbb{E}(X_n^2) < \infty$ ,  $n \geq 1$ , and  $\{b_n, n \geq 1\}$  be a sequence of positive nondecreasing real numbers, then for every  $\varepsilon > 0$

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon\right) \leq \frac{8}{\varepsilon^2} \sum_{k=1}^n \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^n \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}.$$

This result is derived from Theorem 1 by putting  $\phi(x) = |x|^2$  and  $\chi(b) = \varepsilon b$  with  $E(X_n) = 0$ ,  $\mathbb{E}(X_n^2) < \infty$  for all  $n \geq 1$ ,  $u_n = \sum_{i=1}^n X_i^+$  and  $v_n = \sum_{i=1}^n X_i^-$ .

More generally, our results apply for  $\phi(x) = |x|^\nu$  for  $\nu \geq 1$ . This means that we are able to have SLLN's when  $\nu = 1$  that is when the  $X_n$ 's only have finite first moments.

For example, for a sequence of random variables with stable laws of parameters  $1 \leq \alpha_n < 2$ , the second moments are infinite for those random variables while their first moments exists. The results of Amini et al. do not apply.

**4.2. Comparison with other inequalities or SLLN's.** We rediscover Theorem 3.12 of Tómacs *et al.* [18] and Theorem 2.2 of Christophides [2] for demimartingale sequence  $\{S_n, n \in \mathbb{N}\}$  by taking  $\phi(x) = (x)^r$  for all  $x$  nonnegative real and 0 otherwise,  $u_n = (S_n^+)^r$  et  $v_n = (S_n^-)^r$  for all  $r \geq 1$  which are two demisubmartingales and remarking that  $|S_n|^r = (S_n^+)^r + (S_n^-)^r$ .

Our main result also extends Theorem 2.1 and related results of Wang *et al.* [19] in the case where the function  $g$  defined in this above paper is in addition nondecreasing and there exist a positive number real  $K$  such that for all  $(x, y) \in \mathbb{R}^2$ ,

$$g(x + y) \leq K[g(x) + g(y)]$$

for demimartingales random sequences to arbitrary integrale random variables. It is the case for  $g(x) = (x)^\nu$  for all  $x$  nonnegative real and 0 otherwise, for  $\nu \geq 1$

Let us mention that the inequality of Laha and Rohtgi [10] proved for submartingales could be extended to a version of our Theorem 1 for  $\phi(x) = |x|^\nu$ , for  $\nu \geq 1$ .

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